Mathematical Toolkit **Spring 2023**

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1 Orthogonality and orthonormality

Definition 1.1 *Two vectors u, v in an inner product space are said to be* orthogonal *if $\langle u, v\rangle = 0$. A set of vectors $S \subseteq V$ is said to consist of* mutually orthogonal vectors *if $\langle u, v\rangle = 0$ for all $u \neq v$, $u, v \in S$. A set of $S \subseteq V$ is said to be* orthonormal *if $\langle u, v\rangle = 0$ for all $u \neq v$, $u, v \in S$ and $\|u\| = 1$ for all $u \in S$.*

Proposition 1.2 *A set $S \subseteq V \setminus \{0_V\}$ consisting of mutually orthogonal vectors is linearly independent.*

Proposition 1.3 (Gram-Schmidt orthogonalization) *Given a finite set $\{v_1, \ldots, v_n\}$ of linearly independent vectors, there exists a set of orthonormal vectors $\{w_1, \ldots, w_n\}$ such that*

$$\mathrm{Span}\left(\{w_1, \ldots, w_n\}\right) \;=\; \mathrm{Span}\left(\{v_1, \ldots, v_n\}\right) .$$

Proof: By induction. The case with one vector is trivial. Given the statement for k vectors and orthonormal $\{w_1, \ldots, w_k\}$ such that

$$\mathrm{Span}\left(\{w_1, \ldots, w_k\}\right) \;=\; \mathrm{Span}\left(\{v_1, \ldots, v_k\}\right) ,$$

define

$$u_{k+1} \;=\; v_{k+1} - \sum_{i=1}^{k} \langle w_i, v_{k+1}\rangle \cdot w_i \quad \text{and} \quad w_{k+1} \;=\; \frac{u_{k+1}}{\|u_{k+1}\|} .$$

We can now check that the set $\{w_1, \ldots, w_{k+1}\}$ satisfies the required conditions. Unit length is clear, so let's check orthogonality:

$$\left\langle u_{k+1}, w_j\right\rangle = \left\langle v_{k+1}, w_j\right\rangle - \sum_{i=1}^{k} \overline{\langle w_i, v_{k+1}\rangle} \cdot \left\langle w_i, w_j\right\rangle = \left\langle v_{k+1}, w_j\right\rangle - \overline{\left\langle w_j, v_{k+1}\right\rangle} = 0.$$

■

Corollary 1.4 *Every finite dimensional inner product space has an orthonormal basis.*

In fact, Hilbert spaces also have orthonormal bases (which are countable). The existence of a maximal orthonormal set of vectors can be proved by using Zorn's lemma. However, we still need to prove that a maximal orthonormal set is a basis. This follows because we define the basis slightly differently for a Hilbert space: instead of allowing only finite linear combinations, we allow infinite ones. The correct way of saying this is that is we still think of the span as the set of all *finite* linear combinations, then we only need that for any $v \in V$, we can get arbitrarily close to v using elements in the span (a converging sequence of finite sums can get arbitrarily close to its limit). Thus, we only need that the span is *dense* in the Hilbert space V. However, if the maximal orthonormal set is not dense, then it is possible to show that it cannot be maximal. Such a basis is known as a Hilbert basis.

Let V be a finite dimensional inner product space and let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for V. Then for any $v \in V$, there exist $c_1, \ldots, c_n \in \mathbb{F}$ such that $v = \sum_i c_i \cdot w_i$. The coefficients c_i are often called Fourier coefficients. Using the orthonormality and the properties of the inner product, we get $c_i = \langle w_i, v \rangle$. This can be used to prove the following

Proposition 1.5 (Parseval's identity) *Let V be a finite dimensional inner product space and let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for V. Then, for any $u, v \in V$*

$$\langle u, v \rangle = \sum_{i=1}^{n} \langle u, w_i \rangle \cdot \langle w_i, v \rangle .$$

Proof: Just plug in $v = \sum_i \langle w_i, v \rangle w_i$ in the left-hand side and distribute out the inner product. ■

Let's consider $\mathbb{R}^n$. If the w_i are the "standard basis", then this is just writing the inner product $\langle u, v \rangle$ in the usual way as the sum of the products of the coordinate values $\sum_j u_j v_j$ where $u_j = \langle u, w_j \rangle$ and $v_j = \langle v, w_j \rangle$. Parseval's identity says you can do this using any orthonormal basis you want. Plugging in the case of $v = u$, we get $\|u\|^2 = \sum_i u_i^2$.

2 Adjoint of a linear transformation

Definition 2.1 *Let V, W be inner product spaces over the same field $\mathbb{F}$ and let $\varphi : V \to W$ be a linear transformation. A transformation $\varphi^* : W \to V$ is called an* adjoint *of φ if*

$$\langle w, \varphi(v) \rangle = \langle \varphi^*(w), v \rangle \quad \forall v \in V, w \in W .$$

Example 2.2 *Let $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ with the usual inner product, and let $\varphi : V \to W$ be represented by the matrix A. Then φ^* is represented by the matrix A^T. In particular, $\langle w, Av \rangle = w^T Av = (A^T w)^T v = \langle A^T w, v \rangle = \langle \varphi^*(w), v \rangle$. So, a symmetric matrix is "self-adjoint".*

Example 2.3 *Let $V = W = \mathbb{C}^n$ with the inner product $\langle u, v \rangle = \sum_{i=1}^{n} u_i \cdot \overline{v_i}$. Let $\varphi : V \to V$ be represented by the matrix A. Then φ^* is represented by the matrix $\overline{A^T}$.*

Example 2.4 *Let $V = C([0,1], [-1,1])$ with the inner product $\langle f_1, f_2 \rangle = \int_0^1 f_1(x) f_2(x) dx$, and let $W = C([0,1/2], [-1,1])$ with the inner product $\langle g_1, g_2 \rangle = \int_0^{1/2} g_1(x) g_2(x) dx$. Let $\varphi : V \to W$ be defined as $\varphi(f)(x) = f(2x)$. Then, $\varphi^* : W \to V$ can be defined as*

$$\varphi^*(g)(y) = (1/2) \cdot g(y/2) \,.$$

We will prove that every linear transformation has a unique adjoint. However, we first need the following characterization of linear transformations from an inner product space V to the field $\mathbb{F}$ it is over.

Proposition 2.5 (Riesz Representation Theorem) *Let V be a finite-dimensional inner product space over $\mathbb{F}$ and let $\alpha : V \to \mathbb{F}$ be a linear transformation. Then there exists a unique $z \in V$ such that $\alpha(v) = \langle z, v \rangle \ \forall v \in V$.*

We only prove the theorem here for finite-dimensional spaces. However, the theorem holds for any Hilbert space.

Proof: Let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for V. Given v, let $c_1, ..., c_n$ be its Fourier coefficients, so $v = \sum_i c_i w_i$, and $c_i = \langle w_i, v \rangle$. Since α is a linear transformation, we must have $\alpha(v) = \sum_i c_i \alpha(w_i) = \sum_i \langle w_i, v \rangle \alpha(w_i) = \sum_i \left\langle \overline{\alpha(w_i)} w_i, v \right\rangle = \langle z, v \rangle$ for $z = \sum_i \overline{\alpha(w_i)} w_i$. ■

This can be used to prove the following:

Proposition 2.6 *Let V, W be finite dimensional inner product spaces and let $\varphi : V \to W$ be a linear transformation. Then there exists a unique $\varphi^* : W \to V$, such that*

$$\langle w, \varphi(v) \rangle \ = \ \langle \varphi^*(w), v \rangle \quad \forall v \in V, w \in W \,.$$

Proof: For each $w \in W$, the map $\langle w, \varphi(\cdot) \rangle : V \to \mathbb{F}$ is a linear transformation (check!) and hence there exists a unique $z_w \in V$ satisfying $\langle w, \varphi(v) \rangle = \langle z_w, v \rangle \ \forall v \in V$. Consider the map $\beta : W \to V$ defined as $\beta(w) = z_w$. By definition of β,

$$\langle w, \varphi(v) \rangle \ = \ \langle \beta(w), v \rangle \quad \forall v \in V, w \in W \,.$$

To check that β is linear, we note that $\forall v \in V, \ \forall w_1, w_2 \in W$,

$$\langle \beta(w_1 + w_2), v \rangle \ = \ \langle w_1 + w_2, \varphi(v) \rangle \ = \ \langle w_1, \varphi(v) \rangle + \langle w_2, \varphi(v) \rangle \ = \ \langle \beta(w_1), v \rangle + \langle \beta(w_2), v \rangle \,,$$

which implies $\beta(w_1 + w_2) = \beta(w_1) + \beta(w_2)$. $\beta(c \cdot w) = c \cdot \beta(w)$ follows similarly. ■

Note that the above proof only requires the Riesz representation theorem (to define z_w) and hence also works for Hilbert spaces.

3 Self-adjoint transformations

Definition 3.1 *A linear transformation $\varphi : V \to V$ is called* self-adjoint *if $\varphi = \varphi^*$. Linear transformations from a vector space to itself are called* linear operators.

Example 3.2 *The transformation represented by matrix $A \in \mathbb{C}^{n \times n}$ is self-adjoint if $A = \overline{A^T}$. Such matrices are called Hermitian matrices.*

Proposition 3.3 *Let V be an inner product space and let $\varphi : V \to V$ be a self-adjoint linear operator. Then*

- *All eigenvalues of φ are real.*
- *If $\{w_1, \ldots, w_n\}$ are eigenvectors corresposnding to distinct eigenvalues then they are mutually orthogonal.*